

A GAME CHARACTERIZING BAIRE CLASS 1 FUNCTIONS

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ABSTRACT. Duparc introduced a two-player game G_f such that Player II has a winning strategy iff f is Baire class 1. We define a game G'_f for an arbitrary function $f : X \rightarrow Y$ between arbitrary Polish spaces such that Player II has a winning strategy in G'_f iff f is Baire class 1. We also show that G'_f is always determined.

1. INTRODUCTION

A *Polish space* is a separable, completely metrizable topological space. A function $f : X \rightarrow Y$ between Polish spaces X and Y is called *Baire class 1* if the inverse image $f^{-1}(U)$ of an open subset $U \subset Y$ is F_σ in X , that is, it is the countable union of closed sets. In the followings, \mathbb{N} denotes the set $\{0, 1, 2, \dots\}$ of natural numbers, and $\mathbb{N}^\mathbb{N}$ denotes the Baire space, the space of all sequences of natural numbers with the product topology considering \mathbb{N} as discrete.

In his Ph.D. Thesis [5], Wadge introduced the Lipschitz game, in which Player I and Player II play natural numbers alternately starting with Player I. Let us denote by $a_n \in \mathbb{N}$ and $b_n \in \mathbb{N}$ the numbers Player I and Player II play at the n th step of the game, respectively. During a run of the game they build two sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$, where $a, b \in \mathbb{N}^\mathbb{N}$. A strategy τ for Player II defines a function $f_\tau : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$, where $f_\tau(a) = b$ provided that if Player I plays the digits of a then the strategy τ requires Player II to play the digits of b . Then $f : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ is Lipschitz with constant 1 if and only if $f = f_\tau$ for some strategy τ .

The Wadge game, defined also in [5], differs from the Lipschitz game in that Player II can pass at any step of the game. He is only required to play a natural number infinitely many times in each run of the game to make sure he builds an element of the Baire space. Then a strategy τ for Player II still defines a function $f_\tau : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$, and $f : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ is continuous if and only if $f = f_\tau$ for some strategy τ .

In the eraser game introduced by Duparc [3], Player II can not only pass, but he can also erase the last natural number that appears on his board. He is required to build an element $b = (b_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$. In other words, for every $n \in \mathbb{N}$ there has to be an index $m \in \mathbb{N}$ such that the board of Player II contains at least n natural numbers after the m th step of the game, and the first n natural numbers on his board are not erased later. For a function $f : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ let us denote by G_f the eraser game with the following winning condition: Player II wins a run of the game G_f if and only if $f(a) = b$, where $a \in \mathbb{N}^\mathbb{N}$ and $b \in \mathbb{N}^\mathbb{N}$ again denotes the elements

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of the Baire space built by Player I and Player II, respectively. Then $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is Baire class 1 if and only if Player II has a winning strategy in G_f . For a more thorough introduction of the subject, see [4].

In this paper we construct a two-player game G'_f that can be used to characterize Baire class 1 functions between arbitrary Polish spaces. Let X and Y be Polish spaces, let d_X be a compatible, complete metric on X and let $f : X \rightarrow Y$ be an arbitrary function. At the n th step of the game, Player I plays x_n , then Player II plays y_n ,

$$\begin{array}{ccccccc} \text{I} & x_0 & x_1 & x_2 & \dots & & \\ \text{II} & & y_0 & y_1 & y_2 & \dots & \end{array}$$

with the rules that

$$(1) \quad x_n \in X, y_n \in Y \text{ and } d_X(x_n, x_{n+1}) \leq 2^{-n} \text{ for every } n \in \mathbb{N}.$$

From the fact that d_X is complete, it follows that $x_n \rightarrow x$ for some $x \in X$. Player II wins a run of the game if and only if (y_n) is convergent and $y_n \rightarrow f(x)$. We note that if $X = Y = \mathbb{N}^{\mathbb{N}}$ and d_X is the metric with $d_X(x, x') = 2^{-n+1}$, where n is the smallest index with $x(n) \neq x'(n)$, then from a winning strategy of Player II in G_f one can derive a winning strategy for Player II in G'_f and vice versa.

Our main theorem concerning this game is the following.

Theorem 1. *If f is of Baire class 1 then Player II has a winning strategy in G'_f . If f is not of Baire class 1 then Player I has a winning strategy in G'_f . In particular, the game G'_f is determined.*

Remark 2. We note that if we change the rules of the game G'_f and leave out the condition that $d_X(x_n, x_{n+1}) \leq 2^{-n}$ and, of course, change the winning condition so that Player II wins if and only if $(x_n)_{n \in \mathbb{N}}$ is divergent or $y_n \rightarrow f(x)$ where $x_n \rightarrow x$, then Theorem 1 does not remain true. To see this, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(0) = 1$, $f(x) = 0$ if $x \neq 0$. It is easy to see that f is of Baire class 1, but Player I has a winning strategy in this modified game.

We sketch the proof of this. The construction of the winning strategy of Player I is similar to the construction in the proof of Theorem 1. Let us fix a sequence $(x^n)_{n \in \mathbb{N}}$ with $x^n \rightarrow 0$ and $x^{2n} = 0$, $x^{2n+1} \neq 0$ for every $n \in \mathbb{N}$. Now let Player I play a sequence x_0, x_1, \dots with $x_0 = \dots = x_{n_0} = x^0$, $x_{n_0+1} = \dots = x_{n_1} = x^1$ etc., where at each step he waits until Player II plays an element $y_{n_k} \in (3/4, 5/4)$ if k is even, and $y_{n_k} \in (-1/4, 1/4)$ if k is odd. One can easily check that the sequence $(x_n)_{n \in \mathbb{N}}$ is either constant x^k after a while with $y_n \not\rightarrow f(x^k)$, or $x_n \rightarrow 0$ but $|y_{n_k} - y_{n_{k+1}}| \geq 1/2$, hence the sequence $(y_n)_{n \in \mathbb{N}}$ does not converge.

2. PROOF OF THEOREM 1

In this section we prove Theorem 1. We note that Carroy [2] constructs a winning strategy for Player I in G_f (if f is not Baire class 1), and using his ideas one can construct a winning strategy for Player I in G'_f . We include a construction anyway to keep the paper self-contained. The main difficulty is to construct a winning strategy for Player II when f is of Baire class 1. In the eraser game a strategy is constructed using the fact that a Baire class 1 function $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is always the pointwise limit of a sequence of continuous functions. However, this is not the case for functions between arbitrary Polish spaces, hence we need new ideas to complete the proof.

Proof of Theorem 1. Let us fix a compatible, complete metric d_Y for Y and denote the oscillation of f restricted to a closed set $F \subset X$ at a point $x \in F$ by

$$(2) \quad \text{osc}_{f|F}(x) = \inf_{U \ni x \text{ open}} \sup\{d_Y(f(x_1), f(x_2)) : x_1, x_2 \in U \cap F\}.$$

It is easy to check that

$$(3) \quad \text{osc}_{f|F}(x) = 0 \Leftrightarrow f|F \text{ is continuous at } x,$$

and that $\text{osc}_{f|F}$ is upper semi-continuous, hence for every $\varepsilon > 0$,

$$(4) \quad \{x \in F : \text{osc}_{f|F}(x) \geq \varepsilon\} \text{ is closed.}$$

To prove the first assertion of the theorem, suppose that f is of Baire class 1. We need to define a winning strategy for Player II, hence a method of coming up with y_n if x_0, x_1, \dots, x_n are already given. Of course, the possible limit point x of the sequence $(x_n)_{n \in \mathbb{N}}$ is in the closed ball $\overline{B}(x_n, 2^{1-n}) = \{x' \in X : d_X(x', x_n) \leq 2^{1-n}\}$. The idea of the proof is to pick y_n as the image of a point in $\overline{B}(x_n, 2^{1-n})$ at which f behaves “badly”. We note here that for some functions, including the modified Dirichlet function (that is the function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g(p/q) = 1/q$ if p and q are relatively prime and $q > 0$, and $g(x) = 0$ if $x \notin \mathbb{Q}$ or $x = 0$) it would be sufficient to pick y_n as the image of a point in $\overline{B}(x_n, 2^{1-n})$ with the largest oscillation (or a sufficiently large oscillation), because the function restricted to the set of points with large oscillation is continuous. However, in the general case the restriction may not be continuous, and we need to do an iterative construction.

With the help of the set

$$R = \{0\} \cup \{1/n : n \in \mathbb{N}, n > 0\}$$

and the function

$$(5) \quad o_f(F) = \begin{cases} \max\{r \in R : \exists x \in F (\text{osc}_{f|F}(x) \geq r)\} & \text{if } F \neq \emptyset, \\ 0 & \text{if } F = \emptyset, \end{cases}$$

we define a derivative operation on the family of closed subsets of X by

$$D(F) = \begin{cases} \{x \in F : \text{osc}_{f|F}(x) \geq o_f(F)\} & \text{if } o_f(F) > 0, \\ \emptyset & \text{if } o_f(F) = 0. \end{cases}$$

Using (4), $D(F)$ is closed for every closed set $F \subset X$. Using Baire’s theorem that a Baire class 1 function has a point of continuity restricted to every non-empty closed subset (see e.g. [1, Theorem 24.15]), if $o_f(F) > 0$, or equivalently by (3), if $f|F$ is not continuous then $D(F) \subsetneq F$. If $o_f(F) = 0$ but $F \neq \emptyset$ then we also have $D(F) = \emptyset \subsetneq F$, hence

$$(6) \quad F \neq \emptyset \Rightarrow D(F) \subsetneq F.$$

We also note here that

$$(7) \quad F \subset F' \Rightarrow o_f(F) \leq o_f(F'), \text{ and}$$

$$(8) \quad F \neq \emptyset \wedge D(F) = \emptyset \Rightarrow o_f(F) = 0.$$

Now we define the iterated derivative of a closed subset $F \subset X$ the usual way for each $\alpha < \omega_1$, that is,

$$\begin{aligned} D^0(F) &= F, \\ D^{\alpha+1}(F) &= D(D^\alpha(F)), \\ D^\alpha(F) &= \bigcap_{\beta < \alpha} D^\beta(F) \text{ if } \alpha \text{ is limit.} \end{aligned}$$

It can be easily shown by transfinite induction on β that

$$(9) \quad \alpha < \beta \Rightarrow D^\alpha(F) \supset D^\beta(F) \text{ for every closed set } F \subset X.$$

Using (6) and the fact that strictly decreasing transfinite sequences of closed subsets of a Polish space are always countable (see e.g. [1, Theorem 6.9]), for every closed set $F \subset X$ there exists a countable ordinal λ with $D^\lambda(F) = \emptyset$. Let us denote the smallest such λ by $\lambda(F)$.

Now we return to the construction of a winning strategy for Player II. Suppose $x_0, \dots, x_n \in X$ and $y_0, \dots, y_{n-1} \in Y$ are already given, we need to find $y_n \in Y$. Let us use the notation

$$(10) \quad \begin{aligned} X_i^\alpha &= D^\alpha(\overline{B}(x_i, 2^{1-i})), \\ \lambda_i &= \lambda(X_i^0) = \lambda(\overline{B}(x_i, 2^{1-i})), \end{aligned}$$

where, as before, $\overline{B}(x_i, 2^{1-i})$ denotes the closed ball $\{x' \in X : d_X(x', x_i) \leq 2^{1-i}\}$. Note that using the rules of the game (1) and that $x_i \rightarrow x$, we have

$$(11) \quad x \in \overline{B}(x_i, 2^{1-i}) \text{ for every } i.$$

For $i \geq 1$ let γ_i denote the smallest ordinal $\gamma < \omega_1$ such that $o_f(X_i^\gamma) \neq o_f(X_{i-1}^\gamma)$ if such an ordinal exists, and let $\gamma_i = \omega_1$ otherwise. Now we define $y_n \in Y$.

Case (a): $\gamma_n < \omega_1$ and $o_f(X_n^{\gamma_n}) > 0$. In this case, $X_n^{\gamma_n} \neq \emptyset$, so let $y_n \in f(X_n^{\gamma_n})$ be arbitrary.

Case (b): ($\gamma_n = \omega_1$ or $o_f(X_n^{\gamma_n}) = 0$) and λ_n is limit. In this case let $y_n = y_{n-1}$ if $n \geq 1$ and let $y_n \in Y$ be arbitrary otherwise.

Case (c): ($\gamma_n = \omega_1$ or $o_f(X_n^{\gamma_n}) = 0$) and $\lambda_n = \alpha + 1$ for some α . Then let $y_n \in f(X_n^\alpha)$ be arbitrary. This concludes the definition of the strategy for Player II.

In order to show that this strategy is winning for Player II, let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence of legal moves for Player I, that is, $d_X(x_n, x_{n+1}) \leq 2^{-n}$ for every $n \in \mathbb{N}$; we need to show that if Player II follows our strategy then for the sequence $(y_n)_{n \in \mathbb{N}}$ he plays $y_n \rightarrow f(x)$, where x is the limit of $(x_n)_{n \in \mathbb{N}}$.

First of all, we collect a couple of simple properties of the sets X_n^α we will use in our proof. Let $\text{diam}(H)$ denote the diameter of the set $H \subset X$, that is, $\text{diam}(H) = \sup\{d_X(x', x'') : x', x'' \in H\}$.

- Claim 3.** (i) $\forall n (\alpha < \beta \Rightarrow X_n^\alpha \supset X_n^\beta)$,
(ii) $\forall n \forall \alpha (\text{diam}(X_n^\alpha) \leq 2^{2-n})$,
(iii) $\forall n \geq 1 \forall \alpha \leq \gamma_n (\alpha < \omega_1 \Rightarrow X_n^\alpha \subset X_{n-1}^\alpha)$,
(iv) $\forall n \geq 1 (\gamma_n < \omega_1 \Rightarrow o_f(X_n^{\gamma_n}) < o_f(X_{n-1}^{\gamma_n}))$.

Proof. (i) is the application of (9) with $F = X_n^0$.

To see (ii), note that $X_n^\alpha \subset X_n^0$ for every $\alpha < \omega_1$ using (i), hence $\text{diam}(X_n^\alpha) \leq \text{diam}(X_n^0) = \text{diam}(\overline{B}(x_n, 2^{1-n})) \leq 2^{2-n}$.

We prove (iii) by transfinite induction on α . It holds for $\alpha = 0$, since $X_{n-1}^0 = \overline{B}(x_{n-1}, 2^{2^{-n}}) \supset \overline{B}(x_n, 2^{1-n}) = X_n^0$, using that $d_X(x_{n-1}, x_n) \leq 2^{1-n}$ by (1). It is clear that if $X_n^\beta \subset X_{n-1}^\beta$ for every $\beta < \alpha$ then $X_n^\alpha \subset X_{n-1}^\alpha$. It remains to show that if $X_n^\alpha \subset X_{n-1}^\alpha$ and $\alpha + 1 \leq \gamma_n$ then $X_n^{\alpha+1} \subset X_{n-1}^{\alpha+1}$. From $\alpha + 1 \leq \gamma_n$ it follows that $o_f(X_n^\alpha) = o_f(X_{n-1}^\alpha)$. If $o_f(X_n^\alpha) = o_f(X_{n-1}^\alpha) = 0$ then $X_n^{\alpha+1} = X_{n-1}^{\alpha+1} = \emptyset$. Otherwise, $X_n^{\alpha+1} = \{x \in X_n^\alpha : \text{osc}_{f \upharpoonright X_n^\alpha}(x) \geq o_f(X_n^\alpha)\} \subset \{x \in X_n^\alpha : \text{osc}_{f \upharpoonright X_{n-1}^\alpha}(x) \geq o_f(X_n^\alpha)\} \subset \{x \in X_{n-1}^\alpha : \text{osc}_{f \upharpoonright X_{n-1}^\alpha}(x) \geq o_f(X_n^\alpha)\} = X_{n-1}^{\alpha+1}$ using twice the inductive assumption $X_n^\alpha \subset X_{n-1}^\alpha$ and also the fact that $F \subset F'$ implies $\text{osc}_{f \upharpoonright F}(x') \leq \text{osc}_{f \upharpoonright F'}(x')$ for every $x' \in F$.

To see (iv), note that $X_n^{\gamma_n} \subset X_{n-1}^{\gamma_n}$ by (iii), hence $o_f(X_n^{\gamma_n}) \leq o_f(X_{n-1}^{\gamma_n})$ by (7) yielding (iv). \square

In the following lemma we use the notation $d_X(x', F)$ to denote the distance of a point $x' \in X$ and a closed set $F \subset X$, that is, $d_X(x', F) = \inf\{d_X(x', x'') : x'' \in F\}$.

Lemma 4. *Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing sequence of nonempty closed sets with $o_f(F_n) \rightarrow 0$ and $\text{diam}(F_n) \rightarrow 0$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $d_X(x_n, F_n) \rightarrow 0$. Suppose moreover, that $(y_n)_{n \in \mathbb{N}} \subset Y$ is a sequence satisfying that $y_n \in f(F_n)$ for infinitely many n (including $n = 0$), and if $y_n \notin f(F_n)$ then $y_n = y_{n-1}$. Then the limit of $(x_n)_{n \in \mathbb{N}}$ exists, and denoting it by x , we have $y_n \rightarrow f(x)$.*

Proof. It is clear from the facts that $(F_n)_{n \in \mathbb{N}}$ is decreasing sequence of closed sets with $\text{diam}(F_n) \rightarrow 0$ that $\bigcap_n F_n = \{z\}$ for an element $z \in X$. Using also that $d_X(x_n, F_n) \rightarrow 0$ it is clear that $x_n \rightarrow z$, hence the limit x of $(x_n)_{n \in \mathbb{N}}$ indeed exists and $x = z$. Thus $x \in F_n$ for every $n \in \mathbb{N}$. Now let $\varepsilon > 0$, $\varepsilon < 1$ be fixed, we need to find $n_0 \in \mathbb{N}$ with $d_Y(y_n, f(x)) \leq \varepsilon$ for every $n \geq n_0$. Let $n_1 \in \mathbb{N}$ be large enough so that $o_f(F_n) < \varepsilon/2$ for every $n \geq n_1$. Since $x \in F_{n_1}$, it easily follows from the definition of o_f , (5), that $\text{osc}_{f \upharpoonright F_{n_1}}(x) < \varepsilon$, hence for small enough $\delta > 0$,

$$(12) \quad d_X(x', x) \leq \delta \text{ and } x' \in F_{n_1} \text{ imply } d_Y(f(x'), f(x)) < \varepsilon.$$

Now let $n_0 \geq n_1$ be large enough so that

$$(13) \quad \text{diam}(F_n) \leq \delta \text{ for every } n \geq n_0,$$

and since $y_n \in f(F_n)$ for infinitely many n , we can also suppose that

$$(14) \quad y_{n_0} \in F_{n_0}.$$

Let $n \geq n_0$ be fixed, we need to show that $d_Y(y_n, f(x)) < \varepsilon$. If $y_n \in f(F_n)$ then $y_n = f(x')$ for some $x' \in F_n$, hence $d_Y(y_n, f(x)) < \varepsilon$ using (13), (12) and the fact that $n \geq n_1$ implies $F_n \subset F_{n_1}$. If $y_n \notin f(F_n)$ then using (14), there exists $k < n$ with $k \geq n_0$ such that $y_k \in f(F_k)$. Then, as we already saw, $d_Y(y_k, f(x)) < \varepsilon$, moreover, if we choose the largest such k then one can prove easily that the conditions of the lemma imply $y_n = y_k$. Thus the proof of the lemma is complete. \square

We divide the proof of the correctness of the strategy into multiple cases.

Case (1): for infinitely many n , $\gamma_n < \omega_1$. Let

$$(15) \quad \gamma = \min\{\eta : \{n \in \mathbb{N} : \gamma_n \leq \eta\} \text{ is infinite}\}.$$

Since we are in Case (1), $\gamma < \omega_1$.

Case (1a): $\gamma_n \geq \gamma$ for all, but finitely many n . Considering the assumptions of Case (1) and Case (1a), it is easy to see that there exists $m \in \mathbb{N}$ so that

$$(16) \quad \gamma_m = \gamma \text{ and } \gamma_n \geq \gamma \text{ for every } n \geq m.$$

Now we use Lemma 4, with X_{m+n}^γ in place of F_n , y_{m+n} in place of y_n and x_{m+n} in place of x_n . We need to check that the conditions of the lemma hold to complete Case (1a). One can show by induction using (iii) of Claim 3 that $X_j^\gamma \subset X_i^\gamma$ for every $i, j \geq m$ with $i \leq j$, hence $(X_{m+n}^\gamma)_{n \in \mathbb{N}}$ is decreasing. Using this observation and (7) it follows that

$$(17) \quad o_f(X_j^\gamma) \leq o_f(X_i^\gamma) \text{ for every } i, j \geq m \text{ with } i \leq j.$$

Using the definition of γ and the assumption of Case (1a),

$$(18) \quad \gamma_n = \gamma \text{ for infinitely many } n,$$

hence $o_f(X_{m+n+1}^\gamma) < o_f(X_{m+n}^\gamma)$ for infinitely many n by (17). This implies using again (17) and the fact that the range of o_f is R that

$$(19) \quad o_f(X_n^\gamma) > 0 \text{ for every } n \geq m$$

and $(o_f(X_{m+n}^\gamma))_{n \in \mathbb{N}}$ tends to 0. It also follows that $X_{m+n}^\gamma \neq \emptyset$. The fact that $\text{diam}(X_{m+n}^\gamma) \rightarrow 0$ follows from (ii) of Claim 3. Thus the conditions of Lemma 4 concerning only $(X_{m+n}^\gamma)_{n \in \mathbb{N}}$ hold.

To see $d_X(x_{m+n}, X_{m+n}^\gamma) \rightarrow 0$, note that $X_{m+n}^\gamma \subset X_{m+n}^0 = \overline{B}(x_{m+n}, 2^{1-m-n})$ using (i) of Claim 3 and (10). Now it remains to check that the conditions concerning $(y_{m+n})_{n \in \mathbb{N}}$ also hold. Using (18) and (19), y_{m+n} is chosen according to Case (a) for infinitely many n , and using (16), this is also the case for $n = 0$. If y_{m+n} is chosen according to Case (a) then $y_{m+n} \in f(X_{m+n}^{\gamma_n}) \subset f(X_{m+n}^\gamma)$ using (16) and (i) of Claim 3. If y_{m+n} is chosen according to Case (c) then $\lambda_{m+n} = \alpha + 1$ for some $\alpha < \omega_1$ and $y_{m+n} \in f(X_{m+n}^\alpha)$. Moreover, the fact that $X_{m+n}^{\lambda_{m+n}} = \emptyset$ and (8) yields $o_f(X_{m+n}^\alpha) = 0$, hence $\alpha \geq \gamma$ using also (i) of Claim 3, (7) and (19). Thus, $y_{m+n} \in f(X_{m+n}^\alpha) \subset f(X_{m+n}^\gamma)$ using again (i) of Claim 3. If y_{m+n} is chosen according to Case (b) then $n \geq 1$, since for $n = 0$, y_{m+n} is chosen according to Case (a), and $y_{m+n} = y_{m+n-1}$. Thus, the conditions of Lemma 4 indeed hold. The conclusion is exactly what we need, showing that if the game progresses according to Case (1a) then $y_{m+n} \rightarrow f(x)$ as $n \rightarrow \infty$, hence also $y_n \rightarrow f(x)$.

Case (1b): $\gamma_n < \gamma$ for infinitely many n .

Claim 5. *There exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that $\gamma_{n_k} \rightarrow \gamma$ and $\gamma_n > \gamma_{n_k}$ for every $n > n_k$. In particular, the sequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ is also strictly increasing.*

Proof. Let n be arbitrary with $\gamma_n < \gamma$, and let $n_0 = \max\{m \geq n : \gamma_m \leq \gamma_n\}$. The maximum exists using the fact that $\gamma_n < \gamma$ and the definition of γ (15). Now let $n > n_0$ be arbitrary with $\gamma_n < \gamma$ and let $n_1 = \max\{m \geq n : \gamma_m \leq \gamma_n\}$. Iterating this construction we get a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ with the properties that $\gamma_{n_k} < \gamma$ and $\gamma_n > \gamma_{n_k}$ if $n > n_k$. Then $\sup\{\gamma_{n_k} : k \in \mathbb{N}\} \leq \gamma$, and using (15) again, one can easily see that $\gamma_{n_k} \rightarrow \gamma$. \square

Now we fix such a sequence $(n_k)_{k \in \mathbb{N}}$ and use Lemma 4 again, with x_{n_0+n} in place of x_n , y_{n_0+n} in place of y_n , and for $n \in \mathbb{N}$ taking the unique $k \in \mathbb{N}$ with $n_k \leq n_0 + n < n_{k+1}$ we use $X_{n_0+n}^{\gamma_{n_k}}$ in place of F_n . It is easy to check that we defined the set F_n for every $n \in \mathbb{N}$. We now check that the conditions of Lemma 4 hold. To prove that $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence, let $n \in \mathbb{N}$, we need to show that $F_{n+1} \subset F_n$. Let $k \in \mathbb{N}$ be the unique natural number with $n_k \leq n_0 + n < n_{k+1}$. If $n_0 + n + 1 < n_{k+1}$ then $F_{n+1} = X_{n_0+n+1}^{\gamma_{n_k}}$, showing that $F_{n+1} = X_{n_0+n+1}^{\gamma_{n_k}} \subset$

$X_{n_0+n}^{\gamma_{n_k}} = F_n$ using that $\gamma_{n_0+n+1} > \gamma_{n_k}$ provided by Claim 5, and (iii) of Claim 3. If $n_0 + n + 1 = n_{k+1}$ then $F_{n+1} = X_{n_{k+1}}^{\gamma_{n_{k+1}}} \subset X_{n_{k+1}-1}^{\gamma_{n_{k+1}}} = X_{n_0+n}^{\gamma_{n_{k+1}}} \subset X_{n_0+n}^{\gamma_{n_k}} = F_n$ using (iii) and (i) of Claim 3.

Now we show that $o_f(F_n) \rightarrow 0$. Since $(F_n)_{n \in \mathbb{N}}$ is decreasing, it is enough to show that $o_f(F_{n_k-n_0}) \rightarrow 0$ by (7). For every $k \in \mathbb{N}$ one can easily show by induction that $o_f(F_{n_k-n_0}) = o_f(X_{n_k}^{\gamma_{n_k}}) = o_f(X_{n_{k+1}-1}^{\gamma_{n_k}})$, using the properties of $(n_k)_{k \in \mathbb{N}}$ provided by Claim 5 and (iii) of Claim 3. Thus, $o_f(F_{n_k-n_0}) = o_f(X_{n_k}^{\gamma_{n_k}}) = o_f(X_{n_{k+1}-1}^{\gamma_{n_k}}) \geq o_f(X_{n_{k+1}-1}^{\gamma_{n_{k+1}}}) > o_f(X_{n_{k+1}}^{\gamma_{n_{k+1}}}) = o_f(F_{n_{k+1}-n_0})$, using (i) and (iii) of Claim 3. Hence, using also that the range of o_f is R by (5), $o_f(F_{n_k-n_0}) \rightarrow 0$ as $k \rightarrow \infty$, thus $o_f(F_n) \rightarrow 0$. Moreover, we also see that

$$(20) \quad o_f(F_n) > 0 \text{ for every } n,$$

using again that $(F_n)_{n \in \mathbb{N}}$ is decreasing and (7), hence clearly $F_n \neq \emptyset$ for every $n \in \mathbb{N}$. The fact that $\text{diam}(F_n) \rightarrow 0$ follows again from (ii) of Claim 3.

Using that $F_n \subset X_{n_0+n}^0 = \overline{B}(x_{n_0+n}, 2^{1-n_0-n})$, clearly $d_X(x_{n_0+n}, F_n) \rightarrow 0$.

It remains to show that y_{n_0+n} satisfies the conditions of Lemma 4. Let n be fixed and let k be the unique natural number with $n_k \leq n_0 + n < n_{k+1}$. If y_{n_0+n} was chosen according to Case (a) then $y_{n_0+n} \in f(X_{n_0+n}^{\gamma_{n_0+n}}) \subset f(X_{n_0+n}^{\gamma_{n_k}}) = f(F_n)$ using the fact that $\gamma_{n_k} \leq \gamma_{n_0+n}$ by Claim 5 and (i) of Claim 3. If y_{n_0+n} was chosen according to Case (c) then $\lambda_{n_0+n} = \alpha + 1$ for some α with $y_{n_0+n} \in f(X_{n_0+n}^\alpha)$. Using that $X_{n_0+n}^\alpha \neq \emptyset$ and $X_{n_0+n}^{\alpha+1} = X_{n_0+n}^{\lambda_{n_0+n}} = \emptyset$, $o_f(X_{n_0+n}^\alpha) = 0$ by (8). Using also that $o_f(X_{n_0+n}^{\gamma_{n_k}}) = o_f(F_n) > 0$ by (20), we necessarily have $X_{n_0+n}^\alpha \subset X_{n_0+n}^{\gamma_{n_k}} = F_n$ using (i) of Claim 3 and (7). Hence $y_{n_0+n} \in f(F_n)$. If y_{n_0+n} was chosen according to Case (b) then $y_{n_0+n} = y_{n_0+n-1}$. Note also that if $n = 0$, $F_0 = X_{n_0}^{\gamma_{n_0}}$ with $\gamma_{n_0} < \gamma$, hence $\gamma_{n_0} < \omega_1$. Using also (20) one can see that y_{n_0} was chosen according to Case (a), hence for $n = 0$, $y_{n_0} \in f(F_0)$, completing the proof that the conditions of Lemma 4 are satisfied. Then the conclusion of the lemma ensures that $y_n \rightarrow f(x)$.

Note that Case (1a) and Case (1b) covers all subcases of Case (1), hence it remains to show that $y_n \rightarrow f(x)$ even in the following case.

Case 2: $\gamma_n = \omega_1$ for all, but finitely many n . Let $m \in \mathbb{N}$ be large enough so that

$$(21) \quad \gamma_n = \omega_1 \text{ for every } n \geq m.$$

From this fact using also (i) and (iii) of Claim 3 one can easily show first by induction on j and then on β that

$$(22) \quad m \leq i \leq j, \alpha \leq \beta \Rightarrow X_i^\alpha \supset X_j^\beta.$$

It follows easily that if $m \leq i \leq j$ then $\lambda_i \geq \lambda_j$, hence, using that the ordinal numbers are well-ordered there exists $\lambda < \omega_1$ and $M \geq m$ such that

$$(23) \quad n \geq M \Rightarrow \lambda_n = \lambda.$$

Claim 6. $x \in X_n^\alpha$ for every $n \geq M$ and $\alpha < \lambda$.

Proof. Let $\alpha < \lambda$ be fixed throughout the proof. Using (10) and (i) of Claim 3, $\text{diam}(X_n^\alpha) \leq \text{diam}(X_n^0) = \text{diam}(\overline{B}(x_n, 2^{1-n})) \leq 2^{2-n}$, hence using also (22), $(X_n^\alpha)_{n \geq M}$ is a decreasing sequence of closed sets with $\text{diam}(X_n^\alpha) \rightarrow 0$. They are nonempty using (23) and that $\alpha < \lambda$, hence, there is a unique $x^\alpha \in X$ with $\{x^\alpha\} = \bigcap_{n \geq M} X_n^\alpha$. Using again the fact that $X_n^\alpha \subset X_n^0 = \overline{B}(x_n, 2^{1-n})$ and (11),

$d_X(x^\alpha, x) \leq 2^{2-n}$ for every $n \in \mathbb{N}$, hence $x = x^\alpha$. Thus the proof of the claim is complete. \square

It follows from Claim 6 that λ is successor, since otherwise by (23) for any $n \geq M$, $X_n^{\lambda_n} = X_n^\lambda = \bigcap_{\alpha < \lambda} X_n^\alpha \supset \{x\} \neq \emptyset$, contradicting the fact that λ_n is the smallest ordinal with $X_n^{\lambda_n} = \emptyset$. Now let $\lambda = \alpha + 1$, hence

$$(24) \quad \lambda_n = \lambda = \alpha + 1 \text{ for every } n \geq M.$$

Now we use Lemma 4 again to prove that $y_n \rightarrow f(x)$ with X_{M+n}^α in place of F_n , x_{M+n} in place of x_n and y_{M+n} in place of y_n . Now we check that the conditions of the lemma hold. (22) and the fact that $M \geq m$ shows that $(F_n)_{n \in \mathbb{N}} = (X_{M+n}^\alpha)_{n \in \mathbb{N}}$ is a decreasing sequence of closed sets. Since λ_n is the smallest ordinal with $X_n^{\lambda_n} = \emptyset$, $X_n^\alpha \neq \emptyset$ and $D(X_n^\alpha) = X_n^\lambda = \emptyset$ by (24), hence $o_f(X_n^\alpha) = 0$ by (8). The fact that $\text{diam}(F_n) \rightarrow 0$ and $d_X(x_{M+n}, F_n) \rightarrow 0$ follows easily from $F_n = X_{M+n}^\alpha \subset X_{M+n}^0 = \overline{B}(x_{M+n}, 2^{1-M-n})$ by (i) of Claim 3.

Now we show that $y_{M+n} \in F_n = X_{M+n}^\alpha$ for every $n \in \mathbb{N}$. Using (21), (24) and the fact that $M \geq m$, y_{M+n} is chosen according to Case (c), hence $y_{M+n} \in X_{M+n}^\alpha$, showing that the conditions of Lemma 4 are satisfied. The conclusion of the lemma ensures that $y_n \rightarrow f(x)$, completing the analysis of Case (2). Thus, the proof of the first assertion of the theorem is complete.

It remains to show that if f is not of Baire class 1, then Player I has a winning strategy. We use Baire's theorem again, that states that a function is of Baire class 1 if and only if it has a point of continuity restricted to every non-empty closed set (see e.g. [1, Theorem 24.15]). Hence, there is a non-empty closed set $F \subset X$ such that $f \upharpoonright F$ is not continuous at any point of F . Then $\text{osc}_{f \upharpoonright F}(x) > 0$ for every $x \in F$ by (3), hence $F = \bigcup_n F_n$, where

$$F_n = \bigcup \left\{ x \in F : \text{osc}_{f \upharpoonright F}(x) \geq \frac{1}{n} \right\}.$$

Using (4), F_n is closed for every n . Baire's category theorem implies that there exists $n \in \mathbb{N}$ such that F_n is dense in an open portion of F . Let us fix such an n , then using that F_n is closed, there exists $U \subset X$ open with $\emptyset \neq U \cap F = U \cap F_n \subset F_n$. Let C be the closure of $U \cap F_n$ and let $\varepsilon = \frac{1}{n}$. Then $C \subset F_n$. We first show that

$$(25) \quad \text{osc}_{f \upharpoonright C}(x) \geq \varepsilon \text{ for every } x \in C.$$

Indeed, if $x \in U \cap F_n$ then one can easily see that the oscillation of x is independent of the values of f outside U , hence $\text{osc}_{f \upharpoonright C}(x) = \text{osc}_{f \upharpoonright U \cap F_n}(x) = \text{osc}_{f \upharpoonright U \cap F}(x) = \text{osc}_{f \upharpoonright F}(x) \geq \varepsilon$ using that $U \cap F = U \cap F_n$ and that $\varepsilon = \frac{1}{n}$. Now using that $\{x \in C : \text{osc}_{f \upharpoonright C} \geq \varepsilon\}$ is closed by (4), it necessarily contains C , showing (25).

Now we construct a strategy for Player I. Let Player I play an arbitrary element $x_0 = x^0 \in C$. Then Player I plays $x_0 = x_1 = \dots = x^0$ until Player II first plays an elements $y_n \in B(f(x^0), \varepsilon/7)$, where $B(f(x^0), \varepsilon/7)$ denotes the open ball $\{y \in Y : d_Y(y, f(x^0)) < \varepsilon/7\}$. So let n_0 be the smallest natural number with $y_{n_0} \in B(f(x^0), \varepsilon/7)$, if such a number exists. If no such number exists then Player I plays $x_n = x^0$ at every step of the game and Player II plays a sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n \not\rightarrow f(x)$. So we can suppose that at some point, Player II plays $y_{n_0} \in B(f(x^0), \varepsilon/4)$. Then Player I responds with $x_{n_0+1} = x^1 \in B(x^0, 2^{-n_0}) \cap C$ and $d_Y(f(x^1), f(x^0)) \geq \varepsilon \cdot \frac{3}{7}$. Such an element x^1 exists using (25) and the definition of the oscillation, (2).

Now let Player I play x^1 until Player II plays elements outside of the ball $B(f(x^1), \varepsilon/7)$. So let n_1 be the smallest natural number with $y_{n_1} \in B(f(x^1), \varepsilon/7)$, if such a number exists. If no such number exists then Player I plays x^1 indefinitely, with Player II playing a sequence y_n with $y_n \not\rightarrow f(x^1)$. Hence, we can suppose that such an index n_1 exists. Then we note that from $d_Y(f(x^1), f(x^0)) \geq \varepsilon \cdot \frac{3}{7}$, $y_{n_0} \in B(f(x^0), \varepsilon/7)$ and $y_{n_1} \in B(f(x^1), \varepsilon/7)$ it follows that $d_Y(y_{n_0}, y_{n_1}) \geq \varepsilon/7$. Now we pick $x^2 = x_{n_1+1}$ with $x^2 \in B(x^1, 2^{-n_1}) \cap C$ and $d_Y(f(x^2), f(x^1)) \geq \varepsilon \cdot \frac{3}{7}$. Again, the existence of such x^2 is ensured by (25). Iterating the construction, either at some point k , when Player I plays $x^k = x_{n_{k-1}+1} = x_{n_{k-1}+2} \dots$ Player II plays elements $y_{n_{k-1}+1}, y_{n_{k-1}+2}, \dots \notin B(f(x^k), \varepsilon/7)$ and loses, or an infinite sequence $(n_k)_{k \in \mathbb{N}}$ is constructed with $d_Y(y_{n_k}, y_{n_{k+1}}) \geq \varepsilon/7$ for every $k \in \mathbb{N}$, meaning that Player II loses in this case also, finishing the proof of the second assertion of the theorem. Thus, the proof of the theorem is complete. \square

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REFERENCES

- [1] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156, Springer-Verlag, New York, 1995.
- [2] R. Carroy, *Playing in the first Baire class*, MLQ Math. Log. Q. 60 (2014), no. 1-2, 118–132.
- [3] J. Duparc, *Wadge hierarchy and Veblen hierarchy. I. Borel sets of finite rank*, J. Symbolic Logic 66 (2001), no. 1, 56–86.
- [4] L. Motto Ros, *Game representations of classes of piecewise definable functions*, MLQ Math. Log. Q. 57 (2011), no. 1, 95–112.
- [5] W. W. Wadge, *Reducibility and determinateness on the Baire space*, Ph.D. Thesis, University of California, Berkeley, 1983.

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